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A BOUNDARY OF THE SET OF THE RIEMANNIAN MANIFOLDS WITH BOUNDED CURVATURES AND DIAMETERS

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Dedicated to Professor Itiro Tamura on his sixtieth birthday

0. Introduction

In [12], Gromov introduced a metric (Hausdorff distance) on the class of all metric spaces. There, he proved the precompactness of the set consisting of the isometry classes of Riemannian manifolds with bounded curvatures and diameters. In this paper we shall study the structure of the closure of this set.

Definition 0.1. For a natural number n and $D \in (0, \infty]$, we let $\mathscr{M}(n, D)$ denote the set consisting of all isometry classes of compact Riemannian manifolds M such that

(0.2.1) the dimension of M is equal to n,

(0.2.2) the diameter of M is smaller than D,

(0.2.3) the sectional curvature of M is smaller than 1 and greater than -1.

The following problem is fundamental in the study of the Hausdorff distance on $\mathscr{M}(n, D)$.

Problem 0.3. (A) Determine the closure of $\mathscr{M}(n, D)$ with respect to the Hausdorff distance. (Hereafter $\mathscr{CM}(n, D)$ denotes the closure.)

(B) Let X_i ($i = 1, 2, \cdots$) be a sequence of elements of $\mathscr{CM}(n, D)$. Suppose X_i converges to a metric space X with respect to the Hausdorff distance. Then, describe the relation between the topological structures of X_i and X.

Our main result on Problem 0.3(A) is Theorem 0.5 and those on Problem 0.3(B) are Theorems 0.12 and 10.1.

First we deal with Problem 0.3(A). Let $\mathscr{PM}_n$ denote the set of all pointed compact Riemannian manifolds (M, p) satisfying (0.2.1) and (0.2.3), and $\mathscr{CPM}_n$ the closure of $\mathscr{PM}_n$ with respect to the pointed Hausdorff distance (see 1.6). If $M \in \mathscr{CM}(n, D)$ then $(M, p) \in \mathscr{CPM}_n$ for each $p \in M$. We let $M(n, D, \mu)$ denote the set of the elements of $\mathscr{M}(n, D)$ whose injectivity radii

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are greater than μ. Put

$$\operatorname{Int}(\mathscr{M}(n,D)) = \bigcup_{\mu>0} \mathscr{CM}(n,D,\mu),$$
$$\partial\mathscr{M}(n,D) = \mathscr{CM}(n,D) - \operatorname{Int}(\mathscr{M}(n,D)).$$

$\operatorname{Int}(\mathscr{PM}_n)$ and $\partial\mathscr{PM}_n$ are defined similarly.

Gromov, in [12], proved that the elements of $\operatorname{Int}(\mathscr{PM}_n)$ are manifolds. In general, elements of $\partial\mathscr{PM}_n$ have singularities. Several examples of elements of $\partial\mathscr{PM}_n$ can be constructed with help from torus actions and more generally from F-structures (see [3], [18]). One of the main theorems of this paper asserts that every element of $\mathscr{CPM}_n$ is locally of this type. To state it, we need a definition.

Definition 0.4. We say elements (X, p_0) and X of $\mathscr{CPM}_n$ and $\mathscr{CM}(n,\infty)$ are *smooth* if they satisfy the following:

For each point p of X, there exist a neighborhood U of p in X, a compact Lie group G_p and a faithful representation of G_p into the orthogonal group, $\mathrm{O}(n)$, such that the identity component of G_p is isomorphic to a torus and that U is homeomorphic to V/G_p for some neighborhood V of 0 in $\mathbf{R}^m$. Furthermore there exists a G_p-invariant smooth Riemannian metric g on V such that U is isometric to $(V/G_p, \bar{g})$, where $\bar{g}$ denotes the quotient metric.

Theorem 0.5. *Smooth elements are dense in $\mathscr{CPM}_n$ with respect to the pointed Lipschitz distance. In particular, every element of $\mathscr{CPM}_n$ is homeomorphic to a smooth one.*

Theorem 0.5 gives us complete information on the local topological structure of the elements of $\mathscr{CPM}_n$. Our result on global structure is not yet complete.

Theorem 0.6. *Let $X \in \mathscr{CPM}_n$. Then there exists a Riemannian manifold M on which $\mathrm{O}(n)$ acts as isometries such that the following holds.*

(0.7.1) *X is isometric to $M/\mathrm{O}(n)$. (Let $P\colon M \to X$ be the projection.)*

(0.7.2) *For each point p of X the group $\{g \in \mathrm{O}(n) \mid g(p) = p\}$ is isomorphic to G_p, where G_p is as in Definition* 0.4.

By virtue of Theorem 0.5, the Hausdorff dimension of each element of $\mathscr{CPM}_n$ is an integer. Inspecting this fact, we define stratifications on $\mathscr{CPM}_n$ and $\mathscr{CM}(n,D)$ as follows.

Definition 0.8.

$$\Xi\mathscr{M}_k(n,D) = \{X \in \mathscr{CM}(n,D) \mid (\text{Hausdorff dimension of } X) \le n-k\},$$
$$\Xi\mathscr{PM}_{n,k} = \{(X,p) \in \mathscr{CPM}_n \mid (\text{Hausdorff dimension of } X) \le n-k\}.$$

[12, 8.39] implies $\Xi\mathscr{M}_1(n,D) = \partial\mathscr{M}(n,D)$.

Our next result concerns the metric structure of the smooth elements of $\mathscr{CPM}_n$. Let (X, p_0) be a smooth element of $\Xi\mathscr{PM}_{n,k} - \Xi\mathscr{PM}_{n,k+1}$. Then X has a stratification $X = S_0(X) \supset S_1(X) \supset \cdots \supset S_k(X)$ such that $S_i(X) - S_{i+1}(X)$ is a $(k-i)$-dimensional smooth Riemannian manifold. In the case when X is not necessarily smooth, we define a stratification on X using that of a smooth one and the Lipschitz homeomorphism given by Theorem 0.5. [7, Example 1.13] or [16] shows that we cannot obtain an upper bound of the sectional curvatures of $S_i(X) - S_{i+1}(X)$ while X moves on $\mathscr{CPM}_n$. But we have the following.

Theorem 0.9. *Let (X_i, p_i) be a sequence of smooth elements of $\Xi\mathscr{PM}_{n,k} - \Xi\mathscr{PM}_{n,k+1}$ and (X, p_0) a pointed metric space. Assume that (X_i, p_i) converges to (X, p_0) in the sense of the pointed Hausdorff distance. Then X is contained in $\Xi\mathscr{PM}_{n,k+1}$ if one of the following two conditions is satisfied.*

(0.10.1) *There exist a positive c and a positive integer j such that*

(0.10.1.a) *$p_i \in S_j(X_i)$ and $d(p_i, S_{j+1}(X_i)) \geq c$, and*

(0.10.1.b) *the sectional curvatures of $S_j(X_i) - S_{j+1}(X_i)$ at p_i are unbounded.*

(0.10.2.a) *p_i satisfies* (0.10.1.a) *and*

(0.10.2.b) *the injectivity radius of $S_j(X_i) - S_{j+1}(S_i)$ at p_i converges to* 0 *when i tends to infinity.*

Furthermore, in the case when (0.10.1) *holds, we have $p_0 \in S_1(X)$.*

Theorems 0.5 and 0.9, combined with [9], [19] or [12, 8.28], imply the following.

Corollary 0.11. *Let (X, p_0) be a (not necessarily smooth) element of $\mathscr{CPM}_n$. Then $S_k(X) - S_{k+1}(X)$ is a Riemannian manifold with continuous metric tensor and $C^{1,\alpha}$-distance function, where α is an arbitrary number contained in* $[0, 1)$.

Next, we shall describe our results from Problem 0.3(B). In the case when $X_i \in \mathrm{Int}(\mathscr{M}(n, D))$ we have the following:

Theorem 0.12. *Let $M_i \in \mathrm{Int}(\mathscr{M}(n, D))$ and $X \in \mathscr{CM}(n, D)$. Suppose $\lim_{i\to\infty} d_{\mathrm{H}}(M_i, X) = 0$. Then, for each sufficiently large i, there exists a differentiable map $f\colon M_i \to X$ satisfying the following.*

(0.13.1) *For each j, the restriction of f to $f^{-1}(S_j(X) - S_{j+1}(X))$ is a fiber bundle whose fiber is diffeomorphic to an infranilmanifold.*

(0.13.2) *Let $p_0 \in X - S_1(X)$, $p \in X$, $F = f^{-1}(p - 0)$ and G_p be the group given in Definition* 0.4. *Then G_p acts freely on F and $f^{-1}(p)$ is diffeomorphic to the quotient space F/G_p.*

More precise informations on the map f and on its relation to the metric structures of X and M_i are in §10. In the case when $X_i \in \partial\mathscr{M}(n, D)$, we can prove a similar result. But, since the result is a bit complicated, we do not state it here (see §10), and restrict ourselves to the following simple case.

Theorem 0.14. $\Xi\mathscr{PM}_{n,k} - \Xi\mathscr{PM}_{n,k+1}$ *is complete with respect to the pointed Lipschitz distance. The pointed Hausdorff distance and the pointed Lipschitz distance define the same topology on it.*

In the case when $k = 0$, Theorem 0.14 follows from the results of [12].

In the course of the proof of Theorem 0.12, we shall prove the following finiteness theorem.

Theorem 0.15. *For each n and $D < \infty$, there exists a finite set Σ of manifolds whose dimensions are not greater than $n + (n-1)(n-2)/2$ and which satisfy the following. For each element M of $M(n, D)$, there exists a smooth map f from the bundle of orthonormal frames of M to an element of Σ, such that f is a fiber bundle with an infranilmanifold fiber.*

The following result is a direct consequence of Theorem 0.15.

Corollary 0.16. $\sup\{\sum_i \operatorname{rank}(H_i(M;K)) \mid M \in M(n,D), K\colon \text{field}\}$ *is finite for each $D < \infty$ and n.*

By a different method, M. Gromov proved in [11] the same conclusion without assuming that sectional curvature is less than or equal to 1.

The organization of this paper is as follows. In Chapter I, we shall prove Theorem 0.5. In §2, we take an element (X, p_0) of $\mathscr{CPM}_n$ and prove that, to verify Theorem 0.5, it suffices to show that X is smooth if (X, p_0) is a limit of pointed Riemannian manifolds (M_i, p_i), the derivatives of whose curvatures are uniformly bounded. In §3, we shall represent a neighborhood of each point of X as the quotient B/G of a Riemannian manifold B by a smooth action of a Lie group germ G. For this purpose, we shall pull back the metrics of M_i to their tangent spaces $T_{p_i}(M_i)$, following [12, 8.33–8.36], and represent neighborhoods of p_i as the quotient spaces B/Γ_i. Taking the limit, we obtain B and G. In §4, we shall prove that G is nilpotent. The proof of Theorem 0.5 is completed in §5.

Chapter II is devoted to the study of Problem 0.3(B). In §6, we shall introduce the set $\mathscr{FPM}_n$ consisting of the frame bundles of the elements of $\mathscr{PM}_n$, and shall prove that the smooth elements of the closure $\mathscr{CFPM}_n$ are Riemannian manifolds. In §7, we shall give an estimate on the sectional curvatures of the smooth elements of $\mathscr{CFPM}_n$. In §8, we shall prove Theorem 0.15. In §9, we shall prove an equivariant version of the result of [6], which is used in §10 to prove our results on Problem 0.3(B). The proof of Theorems 0.6 and 0.9 is also in §10.

In §1, we gather several notations used in this paper. The reader can skip this section and return there when §1 is explicitly quoted.

Some of the results of this paper were announced without proof in [7]. There we also gave several examples and open problems. See also [3], [4], [5], [6], and [18] for related results, and [8] for an application.

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1. Notation and preliminary considerations

In this section, X and Y denote metric spaces, $p_0 \in X$, $q_0 \in Y$, and M denotes a Riemannian manifold.

Notation 1.1. We put

$$B_D(p_0, X) = \{p \in X \mid d(p_0, p) < D\},$$
$$B(D) = B_D(0, \mathbf{R}^n), \qquad B = B(1).$$

Notation 1.2. Let $C(X, Y)$ denote the set of continuous maps from X to Y. We define a metric d on $C(X, Y)$ by

$$d(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\}.$$

Notation 1.3. Set

$$FM = \{(V_1, \cdots, V_n) \mid (V_1, \cdots, V_n) \text{ is an orthonormal base of the tangent space of a point of } M\}.$$

We define a metric on FM as follows. Let $\pi\colon FM \to M$ be the natural projection. The fiber of π is identified with the orthogonal group $O(n)$. Fix a canonical metric on $O(n)$. For each $q \in FM$, using the Levi-Civita connection, the tangent space $T_q(FM)$ is decomposed into the vertical subspace $T_q(\pi^{-1}\pi(q))$, and the horizontal subspace H_q. We define a metric on $T_q(\pi^{-1}\pi(q))$ using the canonical metric on $O(n)$ and on H_q so that $d\pi\colon H_q \to T_{\pi(q)}(M)$ is an isometry. Also, we let the horizontal and the vertical subspaces be orthogonal. Thus we obtain a metric on FM. The group $O(n)$ acts as isometries on FM, and the quotient space $FM/O(n)$ with the quotient metric is isometric to M.

Notation 1.4. Let γ be a selfisometry of M. Assume that $p \in M$ and that $d(p, \gamma(p))$ is smaller than the injectivity radius of M at p. Let $l\colon [0, t_0] \to M$ denote the minimal geodesic connecting p with $\gamma(p)$. (We assume that l has unit speed.) Let $P\colon T_{\gamma(p)}(M) \to T_p(M)$ denote the parallel transformation along l. We set

$$t_p(\gamma) = t_0 \cdot \dot{l}(0),$$
$$r_p(\gamma)\colon T_p(M) \to T_p(M)\colon V \mapsto P(d\gamma(V)),$$
$$m_p(\gamma)\colon T_p(M) \to T_p(M)\colon V \mapsto P(d\gamma(V)) + t_p(\gamma),$$
$$\|r_p(\gamma)\| = \text{the supremum of the angles between } V \text{ and } r_p(\gamma)(V),$$
$$\|m_p(\gamma)\| = \|r_p(\gamma)\| + \|t_p(\gamma)\|.$$

Notation 1.5. We put

$$\mathscr{M}(n, D \mid C) = \{M \mid M \text{ satisfies (0.2.1), (0.2.2) and the sectional curvature of } M \text{ is smaller than } C \text{ and greater than } -C\}.$$
$$\mathscr{PM}'_n(C) = \{(M,p) \mid M \in \mathscr{M}(n, \infty \mid C)\}.$$

(We do not assume that the elements of $\mathscr{PM}'_n(C)$ are compact.)

Definition 1.6. We recall the definition of the ε-Hausdorff approximation and its pointed version. A (not necessarily continuous) map $f\colon X \to Y$ [resp. $(X, p_0) \to (Y, q_0)$] is said to be an *ε-Hausdorff approximation* [resp. *ε-pointed Hausdorff approximation*] if

(1.7.1) The ε-neighborhood of $f(X)$ contains Y [resp. $B_{1/\varepsilon}(q_0, Y)$].

(1.7.2) For each two elements x, y of X [resp. $B_{1/\varepsilon}(p_0, X)$] we have

$$|d(x,y) - d(f(x), f(y))| < \varepsilon.$$

We define the *Hausdorff distance* [resp. *pointed Hausdorff distance*] $d_{\mathrm{H}}(X, Y)$ [resp. $d_{\mathrm{H}}((X, p_0), (Y, q_0))$] to be the infimum of the positive numbers ε such that there exist ε-Hausdorff approximations [resp. ε-pointed Hausdorff approximations] from X to Y and from Y to X [resp. from (X, p_0) to (Y, q_0) and from (Y, q_0) to (X, p_0)].

Notation 1.8. We let $d_{\mathrm{L}}(X, Y)$ and $d_{\mathrm{L}}((X, p_0), (Y, q_0))$ denote the Lipschitz distance and the equivariant Lipschitz distance, which is defined in [12, Chapitre 3A].

Definition 1.9. Next, we need equivariant versions of the notion of the Hausdorff distance. Let G and H be groups acting as isometries on X and Y respectively. A pair of maps (f, φ), $f\colon (X, p_0) \to (Y, q_0)$, $\varphi\colon G \to H$, is said to be an *ε-pointed equivariant Hausdorff approximation* if the following hold.

(1.10.1) f is an ε-pointed Hausdorff approximation.

(1.10.2) For each $g \in G$ and $x \in X$, we have

$$d(\varphi(g)(f(x)), f(g(x))) < \varepsilon$$

if x and $g(x)$ are contained in $B_{1/\varepsilon}(p_0, X)$, and if $f(x)$, $f(g(x))$ and $\varphi(g)(f(x))$ are contained in $B_{1/\varepsilon}(q_0, Y)$.

Let the *pointed equivariant Hausdorff distance*, $d_{\text{e.H.}}((X, G, p_0), (Y, H, q_0))$, denote the infimum of the numbers ε such that there exist ε-pointed equivariant Hausdorff approximations from (X, G, p_0) to (Y, H, q_0) and from (Y, H, q_0) to (X, G, p_0). The nonpointed version is defined similarly. The equivariant Hausdorff distance defined here is equivalent to that of [5]. Therefore, [5, Theorem 2.1] implies the following:

Lemma 1.11. *If*

$$\lim_{i\to\infty} d_{\text{e.H.}}((X, G, p_0), (Y, H, q_0)) = 0,$$

then

$$\lim_{i\to\infty} d_{\text{H}}((X/G, \bar{p}_0),\ (Y/H, \bar{q}_0)) = 0.$$

Definition 1.12. Suppose that a group G acts on X and Y as isometries. We say a map f from X to Y is an *ε-G-Hausdorff approximation* if $(f, \text{identity})\colon (X, G) \to (Y, G)$ is an ε-equivariant Hausdorff approximation. We define the *G-Hausdorff distance*, $d_{\text{G-}H}(X, Y)$, to be the infimum of the positive numbers ε such that there exist ε- G-Hausdorff approximations from X to Y and from Y to X.

Lemma 1.13. *Let $\mathscr{M}(n, D; G)$ denote the set of pairs (M, χ) of Riemannian manifolds M contained in $\mathscr{M}(n, D)$ and an isometric action χ of G on M. If $D < \infty$, then $\mathscr{M}(n, D; G)$ is precompact with respect to the G-Hausdorff distance.*

We omit the proof, which is an easier half of the argument presented in [5, §3].

CHAPTER 1

SINGULARITIES OF THE ELEMENTS OF THE BOUNDARY

2. Reduction to the case when the differentials of the curvatures are bounded

First we recall the following result. (The symbol d_{L} is as in 1.8.)

Theorem 2.1 (*Bemelmans, Min-Oo & Ruh* [1]). *For each positive number ε and Riemannian manifold $M \in \mathscr{M}(n, \infty)$, there exists a Riemannian manifold $M' \in \mathscr{M}(n, \infty)$ such that*

(2.2.1) $$d_{\text{L}}(M, M') < \varepsilon,$$

(2.2.2) $$\|\nabla^k R(M')\| < C(n, k, \varepsilon).$$

Here the symbol $R(M')$ denotes the curvature tensor, $\|\ \|$ the C^0-norm, and $C(n, k, \varepsilon)$ a positive number depending only on n, k and ε.

Secondly we need the following. (The symbol d_{H} is defined in 1.6.)

Lemma 2.3. *Let X_i, Y_i, X, Y be metric spaces, all of whose bounded subsets are relatively compact. Suppose that*

$$\lim_{i\to\infty} d_{\mathrm{H}}(X_i, X) = 0, \qquad \lim_{i\to\infty} d_{\mathrm{H}}(Y_i, Y) = 0,$$

and that $d_{\mathrm{L}}(X_i, Y_i) \le \varepsilon$. Then we have $d_L(X, Y) \le \varepsilon$.

Proof. We may assume $d_{\mathrm{H}}(X_i, X) < 1/i$ and $d_{\mathrm{H}}(Y_i, Y) < 1/i$. Then there exist $(1/i)$-Hausdorff approximations $\varphi_i \colon X \to X_i$, $\psi_i \colon Y_i \to Y$. On the other hand, since $d_{\mathrm{L}}(X_i, Y_i) \le \varepsilon$, there exist homeomorphisms $f_i \colon X_i \to Y_i$ satisfying

$$e^{-\varepsilon} \le d(f_i(x), f_i(y))/d(x,y) \le e^{\varepsilon} \tag{2.4}$$

for each x, $y \in X_i$.

Next, take a dense countable subset X_0 of X. By a standard diagonal procedure, we may assume, by taking a subsequence if necessary, that $\psi_i f_i \varphi_i(x)$ converges for each $x \in X_0$. Let $f'(x)$ be the limit. Then formulas (1.7.2) and (2.4) imply

$$e^{-\varepsilon} \le d(f'(x), f'(y))/d(x,y) \le e^{\varepsilon} \tag{2.5}$$

for each x, $y \in X_0$. Therefore f' can be extended to a homeomorphism $f \colon X \to Y$ satisfying (2.5). The required inequality $d_{\mathrm{L}}(X, Y) \le \varepsilon$ follows. q.e.d.

Now we start the proof of Theorem 0.5. Let (X, p_0) be an arbitrary element of $\mathscr{CPM}_n$. Then there exists a sequence (M_i', p_i') of elements of $\mathscr{PM}_n$ such that $\lim_{i\to\infty} d_{\mathrm{H}}((X, p_0), (M_i', p_i)) = 0$. Hence, Theorem 2.1 implies that, for each positive number ε, there exists $(M_i(\varepsilon), p_i(\varepsilon)) \in \mathscr{PM}_n$ such that $d_{\mathrm{L}}((M_i(\varepsilon), p_i(\varepsilon)), (M_i', p_i')) < \varepsilon$ and

$$\|\nabla^k R(M_i(\varepsilon))\| < C(n, k, \varepsilon). \tag{2.6}$$

Since $\mathscr{CPM}_n$ is compact [12, 5.3], we may assume, by taking a subsequence if necessary, that $(M_i(\varepsilon), p_i(\varepsilon))$ converges to a metric space $(X(\varepsilon), p_0(\varepsilon))$ with respect to the Hausdorff distance. Then Lemma 2.3 implies $d_{\mathrm{L}}(X, X(\varepsilon)) \le \varepsilon$. Thus, we see that to prove Theorem 0.5 it suffices to show that $X(\varepsilon)$ is a smooth element of $\mathscr{CPM}_n$. The proof of this fact occupies the rest of this chapter. Hereafter we shall write (M_i, p_i) and (X, p_0) instead of $(M_i(\varepsilon), p_i(\varepsilon))$ and $(X(\varepsilon), p_0(\varepsilon))$, for simplicity.

3. Construction of the Lie group germ

Some part of the argument of this and the next sections overlaps with that of [12, 8.30–8.36 and 8.48–8.51]. But, since the argument here is a bit delicate and since the author cannot understand some part of the argument there, he will not omit the overlapped part.

By changing a base point, we see that it suffices to show that a neighborhood of p is smooth. We may assume that $d_{\mathrm{H}}((X,p_0),(M_i,p_i)) < 1/i$. Let $\varphi_i\colon (X,p_0) \to (M_i,p_i)$ denote a $(1/i)$-Hausdorff approximation and $f_i\colon \mathbf{R}^n \to M_i$ the composition of a linear isometry $\mathbf{R}^n \to T_{p_i}(M_i)$ and the exponential map $T_{p_i}(M_i) \to M_i$. By Rauch's comparison theorem (see [15, Chapter VIII, Theorem 4.1]), the map f_i is of maximal rank on the unit ball B (see 1.1). Let g_i $(= g_{i;j,k})\colon B \to \mathbf{R}^{n^2}$ be the Riemannian metric tensor induced by f_i from that of M_i. Formula (2.6) implies that

$$\left\| \frac{\partial^l g_{i;j,k}}{\partial x_{m_1} \partial x_{m_2} \cdots \partial x_{m_l}} \right\| < C_l.$$

It follows that we may assume, by taking a subsequence if necessary, that g_i converges to a C^∞-metric tensor g_0. Hereafter we let d_i $(i = 0,1,2,\cdots)$ denote the distance function associated to g_i and d the ordinary Euclidean distance.

First, we shall construct a local group G of isometries such that a neighborhood of p_0 in X is isometric to U/G for a neighborhood U of 0 in B. The fundamental definitions on local groups are presented in [20, §23D, $\cdots$,N]. There the notion of an action of a local group on a pointed topological space is not defined. But we omit the definition, since it can be defined in an obvious way.

Now, we define the local group G_i as

$$G_i = \{\gamma \in C(B(1/2),B) \mid f_i\gamma = f_i\},$$

where $C(A,B)$ is as in 1.2. The local group structure on G_i is defined as follows: for $\gamma_1,\gamma_2,\gamma_3 \in G_i$, we put $\gamma_1\gamma_2 = \gamma_3$ if the composition $\gamma_1\gamma_2$ is well defined and coincides with γ_3 in a neighborhood of 0. Next, for $p \in B(1/2)$ and $\varepsilon > 0$, we put

$$G_i(p,\varepsilon) = \{f \in G_i \mid d(f(p),p) < \varepsilon\}.$$

Second, we shall take the limit of G_i. Put

$$L = \{f \in C(B(1/2),B) \mid 1/2 \le d_0(f(x),f(y))/d_0(x,y) \le 2 \quad \text{for each } x,y \in B(1/2)\}.$$

Ascoli-Arzela's theorem implies that L is compact. It is well known that the set of closed subsets of a given compact set is compact with respect to the (usual) Hausdorff distance. Therefore, by taking a subsequence if necessary, we may assume that G_i converges to a closed subset G of L. We can define a local group structure on G by a method similar to that for G_i.

Remark that when a local group H acts as isometries on a pointed metric space (Y, p), the isometry type of a neighborhood of $(p \bmod H)$ in the quotient space Y/H is well defined (see [20, §23J]). We shall let this "local metric space" be denoted by $(Y, p)/H$. In our case, $(B(1/2, 0), 0)/G_i$ is isometric to $B_{1/2}(p_i, M_i)$. (Furthermore, in our case, the 1/2-neighborhood of $(0 \bmod G_i)$ is well defined.) This fact, combined with Lemma 1.11, implies that $(B(1/2, 0), 0)/G$ is isometric to $B_{1/2}(p_0, X)$. Let $\pi\colon B(1/2) \to B_{1/2}(p_0, X)$ and $\pi_i\colon B(1/2) \to B_{1/2}(p_i, M_i)$ denote the natural projections.

Third, we shall prove that our local group G is a Lie group germ. This fact follows from the following:

Lemma 3.1. *Suppose a local group G acts effectively on a pointed Riemannian manifold (M, p) as isometries. Assume that G is closed in $C(B_{D/2}(p, M), B_D(p, M))$. Then G is locally isomorphic to a Lie group and its action on (M, p) is smooth.*

Proof. This lemma seems to be known by the experts. But, since it seems that this fact is not proved in the literature, the proof will be given below. Let $\mathfrak{g}'$ be the set of all vector fields ξ such that the following condition holds.

Condition 3.2. There exists a smooth map $\varphi\colon (-\varepsilon, \varepsilon) \to G$ satisfying the following. (Since G is contained in a Frechet manifold $C(B_{D/2}(p, M), B_D(p, M))$, the smoothness of a map from $(-\varepsilon, \varepsilon)$ to G is well defined.)

$$\varphi(0) = \text{identity}, \tag{3.2.1}$$

$$\left.\frac{D\varphi(t)(p)}{dt}\right|_{t=0} = \xi(p). \tag{3.2.2}$$

Now since

$$\left.\frac{D\varphi_1(t)\varphi_2(t)}{dt}\right|_{t=0} = \left.\frac{D\varphi_1(t)}{dt}\right|_{t=0} + \left.\frac{D\varphi_2(t)}{dt}\right|_{t=0}$$

and since

$$\left.\frac{D}{dt^2}(\varphi_1(t)\varphi_2(t)\varphi_1^{-1}(t)\varphi_2^{-1}(t))\right|_{t=0} = \left[\left.\frac{D\varphi_1(t)}{dt}\right|_{t=0}, \left.\frac{D\varphi_2(t)}{dt}\right|_{t=0}\right],$$

it follows that $\mathfrak{g}'$ is a Lie algebra. Let G' be the local set consisting of all one-parameter groups of transformations associated with the elements of $\mathfrak{g}'$. Using the fact that $\mathfrak{g}'$ is a Lie algebra, we can prove easily that G' is a Lie group germ.

Sublemma 3.3. *G' is a sub-local group of G.*

Proof. Suppose that $\xi \in \mathfrak{g}'$ and that $\varphi\colon (-\varepsilon, \varepsilon) \to G$ satisfies Condition 3.2. Let Φ_t denote the one-parameter group of transformations associated with ξ. We shall prove that $\Phi_{t_0} \in G$ for small t_0. Put $\gamma_n = (\varphi(t_0/n))^n$.

Using (3.2.2), we can prove $\lim_{n\to\infty}\gamma_n = \Phi_{t_0}$. On the other hand, since G is closed, it follows that $\Phi_{t_0} \in G$. q.e.d.

Now, to prove Lemma 3.1, it suffices to show the following:

Sublemma 3.4. *G' contains a neighborhood of the identity of G.*

Proof. Suppose that the sublemma is false. Then there exists a sequence of elements γ_i of $G - G'$ which converges to the identity. Here we need a simple trick to make the action of G free. Let FM be as in 1.3. The action of G can be lifted to a free isometric action on FM. Take an element q of FM. Now, by replacing elements γ_i if necessary, we may assume the following:

(3.5) The minimal geodesic l_i connecting q with $\gamma_i(q)$ is perpendicular to the orbit $G'(q)$.

Now, since γ_i converges to the identity map, we may assume, by taking a subsequence if necessary, that there exists a strictly increasing sequence n_i of positive integers such that $\gamma_i^{n_i}$ converges to a nontrivial element γ. Then, fact (3.5) implies that $\gamma \notin G$. On the other hand we have

Assertion 3.6. $\gamma \in G'$.

Proof. For $t \in [0,1]$, we put $\varphi_t = \lim_{i\to\infty} \gamma_i^{[tm_i]}$, where $[c]$ denotes the maximum integer not greater than c. It is easy to see that φ_t is well defined and is a one-parameter group of transformations. It is also easy to see that $\varphi_1 = \gamma$ and $\varphi_t \in G$. Therefore $\gamma \in G'$ as desired. q.e.d.

This is a contradiction. The proof of Sublemma 3.4 is now complete.

4. Nilpotency of the local group G

Lemma 4.1. *The Lie algebra $\mathfrak{g}$ of G is nilpotent.*

Proof. Take a small neighborhood W of the identity in L such that $\|m_p(\gamma)\| < 0.49$ holds for each element γ of $W \cap G$ and $p \in B(1/2)$ (see 1.4 and 1.1). Now Lemma 4.1 follows from the following:

Lemma 4.1. *There exists a neighborhood W' of the identity in W such that the n-hold commutators of the elements of $G_i \cap W'$ are well defined in G and vanish.*

Remark 4.3. This corresponds to [12, 8.50]. In order to prove this lemma following the line described there, we have to overcome the difficulty pointed out in [2, Remark 3.1.6]. But the author cannot do this directly. Instead, we shall use the result of [6], and proceed as follows.

Proof of Lemma 4.2. By the result of §3, we see that there exists a point p in each neighborhood of 0 in B such that $\{\gamma \in g \mid \gamma(p) = p\} = \{1\}$. Hence, a neighborhood V of $\pi(p)$ in $B_{1/2}(p_0, X)$ is a Riemannian manifold. Therefore, by the main theorem of [6], we conclude that, for each sufficiently large i, there exists a fiber bundle $f_i\colon U_i \to V$ from a neighborhood U_i of $\pi_i(p)$ in M_i to V, such that the fiber of f_i is an infranilmanifold. Furthermore, §5 of [6] implies that there exists a positive number ε independent of i such that $G_i(p,\varepsilon)$ is a sub-local group of the fundamental group of the fiber of f_i. (Remark that $G_i(p,\varepsilon)$ coincides with what is called a local fundamental pseudogroup at the beginning of [6, §5].) Moreover, by virtue of the inequality $\|m_p(\gamma)\| < 0.49$, we see that the fundamental group of the fiber of f_i itself is nilpotent, without taking a finite covering (see the argument in [2, Chapter 3]). Hence every n-hold commutator of elements of $G_i(p,\varepsilon)$ vanishes.

On the other hand, it is easy to see that there exists W' such that

$$G_i(p,\varepsilon) \supset W' \cap G_i$$

for every i. This completes the proof.

5. The proof of Theorem 0.5

Let $\mathfrak{g}$ denote the Lie algebra of G and, for $p \in B(1/2)$, put

$$\mathfrak{h}_p = \{\xi \in \mathfrak{g} \mid \xi(p) = 0\}.$$

Lemma 5.1. *$\mathfrak{h}_p$ is contained in the center of $\mathfrak{g}$.*

Proof. (The following argument was suggested to the author by Hisayosi Matumoto.) Let $\xi \in \mathfrak{h}_p$. Since the closure of the one-parameter group of transformations associated with ξ is compact, it follows that the adjoint representation $\mathfrak{g} \to \mathfrak{g}$, $\eta \mapsto [\eta,\xi]$ is semisimple. Therefore, if ξ is not contained in the center, there exists $\eta \in \mathfrak{g} \otimes \mathbf{C}$ such that $[\eta,\xi] = \alpha\eta$ and $\alpha \neq 0$. But, then the Lie subalgebra $\mathbf{C}\xi \oplus \mathbf{C}\eta$ is not nilpotent. This is a contradiction. q.e.d.

The function which carries p to $\dim \mathfrak{h}_p$ is uppersemicontinuous. Hence, there exists a positive number C such that, for each element p of $B(C)$,

$$\dim \mathfrak{h}_p \leq \dim \mathfrak{h}_0. \tag{5.2}$$

Lemma 5.3. *$\mathfrak{h}_p \subseteqq \mathfrak{h}_0$ for each element p of $B(C/6)$.*

Proof. The proof is by contradiction. Take $\xi \in \mathfrak{h}_p - \mathfrak{h}_0$. Let φ_t be the one-parameter group of transformations associated with ξ. Since the closure $\{\varphi_t \mid t \in \mathbf{R}\}$ is compact, we may assume, by replacing ξ if necessary, that φ_1 is the identity. Put

$$A = \{q \in B(1/2) \mid \eta(q) = 0 \text{ for each } \eta \in \mathfrak{h}_0\}.$$

A is totally geodesic because all elements of $\mathfrak{g}$ are Killing vector fields. Since $p \in B(C/6)$ and since $\varphi_t(p) = p$, it follows that

$$d(\varphi_t(0), 0) \leq C/3. \tag{5.4}$$

On the other hand, since $\mathfrak{h}_0$ is contained in the center, we have $\varphi_t(0) \in A$. Now, define a φ_t-invariant function f on $B(C) \cap A$ by

$$f(q) = \int_0^1 d(\varphi_t(0), q)\, dt.$$

Since A is totally geodesic and since $C \leq 1$, it follows that f is a strictly convex function. On the other hand, formula (5.4) implies that

$$f(q) \geq 2C/3 \quad \text{for } q \in \partial B(C), \qquad f(0) \leq C/3.$$

Therefore, f has a unique minimum q_0 on $A \cap B(C)$. Then $\varphi_t(q_0) = q_0$. It follows that $\xi \in \mathfrak{h}_{q_0}$. On the other hand, $\mathfrak{h}_{q_0} \supset \mathfrak{h}_0$. Thus, we conclude $\dim \mathfrak{h}_{q_0} > \dim \mathfrak{h}_0$. This contradicts (5.2). q.e.d.

For a point p of $B(1/2)$, we put

$$H_p = \{\gamma \in G \mid \gamma(p) = p\}$$

and let H'_p denote the component of the identity of H_p.

Lemma 5.5. *There exists a positive number C' such that $H_p \subseteqq H_0$ for each point p of $B(C'/6)$.*

Proof. For a point p of A, put $\chi(p) = \#(H_p/H'_p)$. It is easy to see that $\chi(p)$ is uppersemicontinuous on A. Then there exists a positive number C' such that for each element p of $B(C') \cap A$, we have $\chi(p) \leq \chi(0)$. Now, we shall prove by contradiction that this number C' has the required property. Suppose that $p \in B(C'/6)$ and $\gamma \in H_p - H_0$. Lemma 5.4 and the compactness of H_p imply that there exists a positive integer m such that γ^m is contained in H_0. Put

$$A' = \{p \in B(C') \mid \gamma(p) = p \text{ for each } \gamma \in H_0\}.$$

Define $f' \colon A' \to \mathbf{R}$ by

$$f'(x) = \sum_{i=1}^{m} d(\gamma^i(x), x).$$

f' is γ-invariant, since $\gamma^m(x) = x$. Hence, as in the proof of Lemma 5.4, we can find $q \in B(C') \cap A'$ such that $\gamma(q) = q$. Therefore $H_q \supset H_0 \cup \{\gamma\}$. It follows that $\chi(q) > \chi(0)$. This is a contradiction. q.e.d.

Lemma 5.1 implies that H'_0 is a torus. Hence $(B(C'/6), 0)/H'_0$ is smooth. Since H_0 is compact, H_0/H'_0 is a finite group. Therefore, $(B(C'/6), 0)/H_0$ is also smooth. Furthermore, using Lemma 5.5, we can prove that H_0 is normalized by G_0. Therefore, G_0H_0/H_0 acts on $(B(C'/6), 0)/H_0$. Then Lemma 5.5 immediately implies that the action of $G_0 \cdot H_0/H_0$ on $B(C'/6)/H_0$ is free. It follows that $(B(C'/6), 0)/H_0G_0$ is smooth. Next, we need the following:

Lemma 5.6. *There exists D such that $G(0, D)$ is contained in H_0G_0.*

Proof. Suppose that there exists a sequence γ_i of elements of G such that $\gamma_i \in G(0,1/i) - H_0G_0$. By taking a subsequence if necessary, we may assume that γ_i converges to an element γ. Then $\gamma(0) = 0$. Therefore $\gamma \in H$. On the other hand, $\lim_{i\to\infty} \gamma^{-1}\gamma_i = 1$. Hence $\gamma^{-1}\gamma_i \in G_0$ for sufficiently large i. Therefore, $\gamma_i \in H_0G_0$. This is a contradiction. q.e.d.

Lemma 5.6 implies that $B_D(p, X)$ is isometic to $(B(D), 0)/H_0G_0$. This completes the proof of Theorem 0.5.

CHAPTER 2

GENERALIZED FIBER BUNDLE THEOREM

6. A compactification of the set of frame bundles

In this chapter, we deal with Problem 0.3(B). One of the difficulties of this problem lies in the fact that the metric space X there is not necessarily a manifold. To avoid this difficulty, we consider the frame bundles. We put

$$\mathcal{FM}(n, D) = \{FM \mid M \in \mathcal{M}(n, D)\},$$
$$\mathcal{FPM}_n = \{(FM, p) \mid M \in M(n, \infty)\}.$$

(The Riemannian manifold FM is defined in 1.3.) Let $\mathcal{CFM}(n, D)$ and $\mathcal{CFPM}_n$ denote the closures of $\mathcal{FM}(n, D)$ and $\mathcal{FPM}_n$ with respect to the Hausdorff distance and the pointed Hausdorff distance respectively. By virtue of the results presented in [17], there exist positive numbers $C_1(n)$ and $C_2(n)$ depending only on n such that

$$\mathcal{FM}(n, D) \subset \mathcal{M}(n + (n-1)(n-2)/2, D + C_1(n) \mid C_2(n))$$

and $\mathcal{FPM}_n \subset \mathcal{PM}_n(C_2(n))$ (see 1-5). It follows that $\mathcal{CFM}(n, D)$ and $\mathcal{CFPM}_n$ are compact. Now, the main result of this and the next sections is the following:

Theorem 6.1. *There exists a positive constant $C_3(n)$ depending only on n such that the intersection of $\mathcal{CFPM}_n$ with*

$$\bigcup_{k=0}^{n+(n-1)(n-2)/2} \mathcal{PM}_k(C_3(n))$$

is dense in $\mathcal{CFPM}_n$ with respect to the pointed Lipschitz distance.

Proof. Let (X, q_0) be an arbitrary element of $\mathcal{CFPM}_n$. Take a sequence of elements (FM_i, q_i) of $\mathcal{FPM}_n$ such that $\lim_{i\to\infty} d_{\mathrm{H}}((FM_i, q_i), (X, q_0)) = 0$. Let $\pi_i \colon FM_i \to M_i$ denote the natural projection. Put $p_i = \pi_i(q_i)$. By an argument similar to one in §2, we may assume, by taking a subsequence if necessary, that

$$\|\nabla^k R(M_i)\| \le C_k.$$

In this section, we shall prove that, in that case, X is a Riemannian manifold. And, in the next section, we shall give an estimate on the sectional curvature of X. It suffices to show this in a neighborhood of q_0.

First remark that we may assume, by taking a subsequence if necessary, that (M_i, p_i) converges to a pointed metric space (Y, p_0) with respect to the pointed Hausdorff distance. We may assume that $d_{\mathrm{H}}((M_i,p_i),(Y,p_0)) < 1/i$ and $d_{\mathrm{H}}((FM_i,q_i),\ (X,q_0)) < 1/i$. Let $\psi_i\colon (X,q_0) \to (FM_i,q_i)$ and $\varphi_i\colon (Y,p_0) \to (M_i,p_i)$ be $(1/i)$-pointed Hausdorff approximations.

Next, we recall the argument of §3. There we defined pairs $((B(1/2),g_i),G_i)$ and $((B(1/2),g_0),G)$ such that $B(1/2)/G_i$ and $B(1/2)/G$ are isometric to $B_{1/2}(p_i,M_i)$ and $B_{1/2}(p_0,X)$ respectively and that G is locally isomorphic to a Lie group.

Now, we can lift the isometric actions of G_i and G on $(B(1/2),g_i)$ and $(B(1/2),g_0)$ to those on $(FB(1/2),\tilde g_i)$ and $(FB(1/2),\tilde g_0)$ respectively, where $\tilde g_i$ and $\tilde g_0$ denote the Riemannian metric defined in 1.3. Since the action of G on $B(1/2)$ is isometric, it follows that the action of G on $FB(1/2)$ is free. Hence $FB(1/2)/G$ is a Riemannian manifold.

On the other hand, it is easy to see that

$$\lim_{i\to\infty} d_{\mathrm{e.H.}}((((FB(1/2),\tilde g_i),G_i,0),((FB(1/2),\tilde g_0),G,0)) = 0.$$

(The symbol $d_{\mathrm{e.H.}}$ is defined in 1.9.) Hence, Lemma 1.11 implies that

$$\lim_{i\to\infty} d_{\mathrm{H}}(FB(1/2)/G_i, FB(1/2)/G) = 0.$$

On the other hand, it is easy to see that $FB(1/2)/G_i$ is isometric to a neighborhood of q_i in FM_i. Therefore $FB(1/2)/G$ is isometric to a neighborhood of q_0 in X. Thus X is a Riemannian manifold, as required.

7. An estimate on sectional curvatures

We begin by proving a lemma.

Notation 7.1. Let G be a local group of isometries acting freely on a pointed Riemannian manifold (M,p). We put

$$(r/t)_p(G) = \sup\{\|r_p(g)\|/d(g(p),p) \mid g \in G, g \ne 1, \quad r_p(g) \text{ is well defined}\}.$$

(The symbol $r_p(g)$ is defined in 1.4.)

Lemma 7.2. *Suppose that the sectional curvature of M is not greater than a and not smaller than b. Then the sectional curvature of M/G at $P(p)$ is not greater than $a + 6((r/t)_p(G))^2$ and not smaller than b, where $P\colon M \to M/G$ denotes the natural projection.*

Proof. Put $q = P(p)$. Let λ be an arbitrary plane contained in $T_q(M/G)$. Take the plane Λ in $T_p(M)$ such that $dP(\Lambda) = \pi$ and Λ is perpendicular to the orbit $G(p)$. Let K_Λ and K_λ denote the sectional curvatures. For $\xi \in \Lambda$ and $t \in \mathbf{R}$, we see easily that

$$P(\exp(t\xi)) = \exp(t(dP(\xi))). \tag{7.3}$$

Now, let $i\colon S^1 \to \Lambda$ be the isometry onto the unit sphere. Recall the following formula.

$$\int_0^t l(\exp(s \cdot i))\, ds = \pi t^2 - \pi K_\Lambda t^4/12 + O(t^5), \tag{7.4}$$

where $l(\exp(t \cdot i))$ denotes the length of the loop, $\theta \mapsto \exp(t \cdot i(\theta))$. Similarly, using (7.3), we see that

$$\int_0^t l(P(\exp(s \cdot i)))\, ds = \pi t^2 - \pi K_\lambda t^4/12 + O(t^5). \tag{7.5}$$

Now, let $\varphi(\theta_0, t)$ denote the angle between

$$\left.\frac{D\exp(t \cdot i(\theta))}{d\theta}\right|_{\theta=\theta_0} \quad \text{and} \quad T_{\exp(t\cdot i(\theta_0))}(G(\exp(t \cdot i(\theta_0)))).$$

Then, it is easy to see that

$$1 \geq \frac{l(P(\exp(t \cdot i)))}{l(\exp(t \cdot i))} \geq \inf\{\sin(\varphi(\theta, t)) \mid \theta \in S^1\}. \tag{7.6}$$

On the other hand, by the definition of $(r/l)_p(G)$, we have

$$\limsup_{t \to 0} \frac{1}{t^2}[1 - \inf\{\sin\varphi(\theta, t) \mid \theta \in S^1\}] \leq \frac{((r/l)_p(G))^2}{2}. \tag{7.7}$$

Now, by (7.4), (7.6) and (7.7), we have

$$\begin{aligned} &\pi t^2 - \pi t^4 K_\Lambda/12 + O(t^5) \\ &\quad\geq \int_0^t l(P(\exp(s \cdot i)))\, ds \\ &\quad\geq \pi t^2 - \pi t^4 K_\Lambda/12 - \pi t^4((r/l)_p(G))^2/2 - O(t^5). \end{aligned}$$

From this formula and formula (7.5), the lemma follows immediately. q.e.d.

Next we shall prove the following:

Lemma 7.8. *Let (M_i, p_i) be a sequence of elements of $\mathcal{CPM}_n$ converging to a smooth element (X, p_0) of $\mathcal{CPM}_n$. Suppose that the sectional curvatures* *of M_i at p_i are unbounded. Then the dimension of the group G_{p_0} in Definition* 0.4 *is positive.*

Proof. Let $(M_{i,j}, p_{i,j})$ be elements of $\mathscr{PM}_n$ such that

$$d_{\mathrm{H}}((M_{i,j}, p_{i,j}), (M_i, p_i)) < 1/j.$$

As in §2, we may assume $\|\nabla^k R(M_{i,j})\| < C_k$. Hence, by the method of §3, we can construct metrics $g_{i,j}$, g_i, g_0 on B and local groups $G_{i,j}$, G_i, G consisting of isometries of $(B(1/2), g_{i,j})$, $(B(1/2), g_i)$, $(B(1/2), g_0)$, such that the quotient spaces $B(1/2)/G_{i,j}$, $B(1/2)/G_i$, $B(1/2)/G$ are isometric to neighborhoods of $p_{i,j}$, p_i, p_0, respectively. Then, Lemma 7.2 implies that the sectional curvatures of M_i at p_i are not smaller than -1 and not greater than $1 + 6 \cdot ((r/t)_0(G_i))^2$. Therefore, by assumption, we see that the numbers $(r/t)_0(G_i)$ are unbounded. Hence, by taking a subsequence if necessary, we may assume that there exists a sequence $\gamma_i \in G_i$ such that $\lim_{i\to\infty} \|r_0(\gamma_i)\|/d(0, \gamma_i(0)) = \infty$. It follows that we can find a sequence of integers n_i such that $\lim_{i\to\infty} d(\gamma_i^{n_i}(0), 0) = 0$, $\lim_{i\to\infty} r_0(\gamma_i^{n_i}) = A$, and that $\lim_{i\to\infty} n_i = \infty$, where $A \in \mathrm{O}(n)$ is a nontrivial element. Now for each number t contained in [0,1], we put $\eta_t = \lim_{i\to\infty} \gamma_i^{[tn_i]}$. Then, $\eta_t \in G$, $\eta_{t_1}\eta_{t_2} = \eta_{t_1+t_2}$, $\eta_1 \neq 1$ and $\eta_t(0) = 0$. Therefore, the dimension of G_p $(= \{g \in G \mid g(0) = 0\})$ is positive. q.e.d.

Now, Theorem 6.1 follows immediately from Lemma 7.8 and the fact that the elements of $\mathscr{CPFM}_n$ are manifolds, which was proved in §6.

8. The proof of Theorem 0.15

We begin by proving a lemma. Put

$$\mathscr{CFM}_k(n, D) = \{M \in \mathscr{CFM}(n, D) \mid \dim M \le n + (n-1)(n-2)/2 - k\},$$

$$\mathscr{CFPM}_{n,k} = \{(M, p_0) \in \mathscr{CFPM}_n \mid \dim M \le n + (n-1)(n-2)/2 - k\}.$$

Lemma 8.1. *For each ε there exists a positive number $\mu(\varepsilon, n)$ such that if a smooth pointed Riemannian manifold $(M, p_0) \in \mathscr{CFPM}_{n,k}$ satisfies $d_H((M, p_0), \mathscr{CFPM}_{n,k+1}) > \varepsilon$, then the injectivity radius of M at p_0 is greater than μ.*

Proof. The proof is by contradiction. Assume that a sequence of pointed Riemannian manifolds $(M_i, p_i) \in \mathscr{CFPM}_{n,k}$ satisfies $d_{\mathrm{H}}((M_i, p_i), \mathscr{CFPM}_{n,k+1}) > \varepsilon$ and that the injectivity radius of M_i at p_i is smaller than $1/i$. By virtue of the compactness of $\mathscr{CFPM}_n$, we may assume, by taking a subsequence if necessary, that (M_i, p_i) converges to an element (X, p_0) of $\mathscr{CFPM}_n$. Then, since the absolute values of sectional curvatures of M_i are bounded, [12, 8.39] implies that the Hausdorff dimension of X is strictly smaller than that of M_i. But, since $d_{\mathrm{H}}((M_i, p_i), \mathscr{CFPM}_{n,k+1}) > \varepsilon$, it follows that $X \notin \mathscr{CFPM}_{n,k+1}$. This is a contradiction.

Proposition 8.2. *There exist positive numbers $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ depending only on n such that the following holds.*

Suppose

$$X \in \mathscr{CFM}_k(n, D), Y \in \mathscr{CFM}(n, D) - \mathscr{CFM}_k(n, D),$$

and

$$d_{\mathrm{H}}(X, \mathscr{CFM}_{k+1}(n, D)) > \varepsilon_{k+1}.$$

Assume, furthermore, that $d_{\mathrm{H}}(X, Y) < \varepsilon_k$.

Then, there exists a map $f\colon X \to Y$ satisfying the following:

(8.3.1) *f is a fiber bundle with an infranilmanifold fiber.*

(8.3.2) *f is an almost Riemannian submersion. Namely, if $\xi \in T_p(M)$ is perpendicular to a fiber of f, then we have*

$$e^{-\tau(d_{\mathrm{H}}(X,Y))} < \|df(\xi)\|/\|\xi\| < e^{\tau(d_{\mathrm{H}}(X,Y))},$$

where $\tau(c)$ is a positive number depending only on c, n and D and satisfying $\lim_{c\to 0} \tau(c) = 0$.

Proof. This is an easy consequence of Theorem 6.1, Lemma 8.1 and the main theorem of [6].

Proof of Theorem 0.15. Define the subsets $\mathscr{U}_k$ of $\mathscr{CFPM}_k(n, D)$ by a downward induction on k as follows.

$$\mathscr{U}_{n+(n-1)(n-2)/2} = \mathscr{CFM}_{n+(n-1)(n-2)/2}(n, D),$$

$$\mathscr{U}_k = \mathscr{CFM}_k(n, D) - \bigcup_{i>k} \{X \in \mathscr{CFM}_k(n, D) \mid d_{\mathrm{H}}(X, \mathscr{U}_i) < \varepsilon_i\}.$$

(Remark that $\mathscr{CFM}_k(n, D)$ is empty for $k > n+(n-1)(n-2)/2$.) Then Lemma 8.1 implies that there exists a positive number μ such that the injectivity radii of the elements of $\bigcup \mathscr{U}_k$ are greater than μ. This fact, combined with Theorem 6.1, the compactness of $\mathscr{U}_k$ and [12, 8.25], implies that there exists a finite set Σ of manifolds such that every element of $\bigcup \mathscr{U}_k$ is diffeomorphic to an element of Σ.

Now, let M be an arbitrary element of $FM(n, D)$. Then, by the definition of $\mathscr{U}_k$, we see that either FM is contained in $\mathscr{U}_0$ or there exist k and $X \in \mathscr{CFM}_k$ such that $d_{\mathrm{H}}(FM, X) < \varepsilon_k$ and $d_{\mathrm{H}}(X, \mathscr{CFM}_{k+1}) > \varepsilon_{k+1}$. In the former case, FM is diffeomorphic to an element of Σ. In the later case, Proposition 8.2 implies that there exists a map $f\colon FM \to X$ satisfying conditions (8.3.1) and (8.3.2), and that X is diffeomorphic to an element of Σ. The proof of Theorem 0.15 is now complete.

9. Equivariant fiber bundle theorem

To deduce Theorem 0.12 from Theorem 6.1, we need the following equivariant version of the result of [6]. (The symbol $d_{G\text{-H}}$ is defined in 1.12.)

Theorem 9.1. *Let G be a locally compact group and let n, μ be positive numbers. Then there exists a positive number $\varepsilon(n,\mu)$ depending only on n and μ and satisfying the following.*

Suppose M, N are Riemannian manifolds on which G acts as isometries. Assume $d_{G\text{-H}}(M,N) < \varepsilon$, $M \in \mathscr{M}(n_1,\infty)$, $N \in \mathscr{M}(n_2,\infty,\mu)$, n_1, $n_2 \leq n$. Then there exists a G-map $f\colon M \to N$ satisfying (8.3.1) *and* (8.3.2).

Proof. There are two methods to prove this result. The first one is to construct f using the result of [6] and to make it a G-map using the center of mass technique (see [13]). The second one is the combination of the methods of [6] and [5, §7]. Here we shall give a proof following the second line. By assumption, we have an ε-G-Hausdorff approximation $\varphi'\colon M \to N$ (see 1.6). We can modify this map and we can assume that φ is a measurable map.

Secondly we use a Hilbert space version of the technique of [12], [14] or [6, §1]. Let $h\colon \mathbf{R} \to [0,1]$ be a function satisfying [6, Condition (1.3)]. And let $L^2(N)$ denote the Hilbert space consisting of all L^2-functions on N. The group G acts on $L^2(N)$ in an obvious way. Define $f_N\colon N \to L^2(N)$ and $f'_M\colon M \to L^2(N)$, $f_M\colon M \to L^2(N)$, by

$$(f_N(p))(q) = h(d(p,q)),$$

$$(f'_M(p))(q) = h\left(\int_{x\in B_\varepsilon(\varphi(q),M)} d(p,x)\,dx/\mathrm{Vol}(B_\varepsilon(\varphi(q),M))\right),$$

$$f_M(p)(q) = \int_{g\in G} f'_M(g(p))(g(q))\mu_G(g),$$

where μ_G denotes the Haar measure. Then, by a method similar to [6], we can prove the following.

(9.2.1) f_N is an embedding.

(9.2.2) Put

$$B_C(Nf_N(N)) = \{(p,u) \in \text{ the normal bundle of } f_N(N) \mid \|u\| < C\}.$$

Then the restriction of the exponential map to $B_C(Nf_N(N))$ is a diffeomorphism, where C is a positive number depending only on n and μ.

(9.2.3) f_M is of C^1-class.

(9.2.4) The image of f_M is contained in the 6ε-neighborhood of $f_N(N)$.

(9.2.5) f_M is transversal to the fibers of the normal bundle of $f_N(N)$. (Here we identify the tubular neighborhood to the normal bundle.)

(9.2.6) f_M and f_N are G-maps.

Now, we put $f = f_N^{-1} \circ \pi \circ \mathrm{Exp}^{-1} \circ f_M$. Facts (9.2.2) and (9.2.4) imply that f is well defined. Fact (9.2.3) implies that f is of C^1-class. Fact (9.2.6) implies that f is a G-map. Fact (9.2.5) implies that f is a fiber bundle. The rest of the proof is similar to [6, §§4 and 5], and hence is omitted. The proof of Theorem 9.1 is now complete.

10. The proof of Theorem 0.12

Our result from Problem 0.3(B) in the case when X is general is the following.

Theorem 10.1. *Let X_i be a sequence of elements of $\mathscr{CM}(n,D)$. Suppose X_i converges to a metric space X with respect to the Hausdorff distance. Then, for sufficiently large i, there exist a map $f\colon X_i \to X$, metric spaces Y_i and Y on which $\mathrm{O}(n)$ acts as isometries and an $\mathrm{O}(n)$-map $\tilde{f}\colon Y_i \to Y$, such that the following holds.*

(10.2.1) *X_i and X are isometric to $Y_i/\mathrm{O}(n)$ and $Y/\mathrm{O}(n)$, respectively. (We let $\pi_i\colon Y_i \to X_i$, $\pi\colon Y \to X$ denote natural projections.)*

(10.2.2) *Y_i and Y are Riemannian manifolds with continuous metric tensors and $C^{1,\alpha}$-distance function.*

(10.2.3) *$\tilde{f}$ satisfies conditions* (8.3.1) *and* (8.3.2).

(10.2.4) *Let $p_i \in Y_i$, $p \in Y$. Then $\{g \in \mathrm{O}(n) \mid g(p) = p\}$ is isomorphic to $G_{\pi(p)}$ (which is defined in* 0.4*), and similarly for p_i.*

(10.2.5) *$f \circ \pi_i = \pi \circ f$.*

Theorems 0.12 and 0.14 are direct consequences of Theorem 10.1. Theorem 0.7 follows immediately from Theorem 10.1, Lemma 7.8 and [12, 8.39].

Proof of Theorem 10.1. Take $\mathscr{M}_{i,j} \in \mathscr{M}(n,D)$ satisfying $d_{\mathrm{H}}(M_{i,j}, X_i) < 1/j$. Lemma 1.13 implies that, by taking a subsequence if necessary, we may assume that

$$d_{\mathrm{O}(n)\text{-}\mathrm{H}}(FM_{i,j}, FM_{i',j'}) < 1/\min(j,j') + 1/\min(i,i').$$

Therefore, there exist $Y_i, Y \in \mathscr{FM}(n,D)$ on which $\mathrm{O}(n)$ acts as isometries such that

$$d_{\mathrm{O}(n)\text{-}\mathrm{H}}(FM_{i,j}, Y_i) < 1/j, \qquad d_{\mathrm{O}(n)\text{-}\mathrm{H}}(Y_i, Y) < 1/i. \tag{10.3}$$

Theorem 6.1, combined with [9], implies that Y_i and Y satisfy (10.2.2). Inequality (10.3), combined with Lemma 1.11, implies (10.2.1). Theorem 9.1 implies that there exists an $\mathrm{O}(n)$-map $\tilde{f}\colon Y_i \to Y$ satisfying (10.2.3). Hence, there exists $f\colon X_i \to X$ satisfying (10.2.5). It is easy to verify (10.2.4). The proof of Theorem 10.1 is now complete.

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